Clarkson Institute Summer School:
Examples Sheet 6 - Maps between groups

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Homomorphisms

1. (a) Find all the homomorphisms $f : C_6 \to C_2$ and $f : C_6 \to C_3$.
   (b) Find all the homomorphisms $f : D_3 \to C_2$ and $f : D_3 \to C_3$.

2. Find all homomorphisms $f : D_n \to C_m$.

3. We say that a map $f : A \to B$ is *surjective* if for every element $b \in B$, there is some element $a \in A$ such that $f(a) = b$. Show that if $f : D_3 \to G$ is a surjective homomorphism, then G must be isomorphic to one of D_3, C_2 or C_1.

4. We say that a map $f : A \to B$ is *injective* if $f(a) = f(a')$ implies $a = a'$; that is, no two distinct elements in A map to the same element in B. Show that $f : G \to H$ is an injective homomorphism between the groups G, H if and only if the kernel $\mathrm{Ker}(f)$ is trivial.

5. Let $\mathrm{Cl}(x) \subseteq G$ be the conjugacy class of an element $x \in G$ of a group G. Suppose that $f : G \to H$ is a homomorphism. Show that $f(\mathrm{Cl}(x)) \subseteq \mathrm{Cl}(f(x))$. Must this hold as an equality?

The isomorphism theorem

6. Define the sets of matrices:

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \ : \ a, b \in \mathbb{C}, a \neq 0 \right\}, \qquad H = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \ : \ a \in \mathbb{C}, a \neq 0 \right\}.$$

   (a) Show that G is a group, and show that $H \leq G$ is a subgroup of G.
   (b) Prove that the map $f : G \to H$ given by:

$$f\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

   is a homomorphism, and hence identify a non-trivial normal subgroup K of G.
   (c) Show that $G/K \cong \mathbb{C}^*$.

7. By identifying an appropriatenormal subgroup $K \trianglelefteq T$, find a non-trivial homomorphism $f : T \to C_3$, where T is the chiral tetrahedral group.

8. Let $SL(n, \mathbb{R})$ be the set of all invertible $n \times n$ matrices whose determinant is equal to one. By finding an appropriate homomorphism $f : GL(n, \mathbb{R}) \to \mathbb{R}*$, show that $SL(n, \mathbb{R}) \leq GL(n, \mathbb{R})$ is a normal subgroup of $GL(n, \mathbb{R})$, and that $GL(n, \mathbb{R})/SL(n, \mathbb{R}) \cong \mathbb{R}^*$.

Automorphisms

9. Identify standard groups to which $\mathrm{Aut}(D_2)$ and $\mathrm{Aut}(D_3)$ are isomorphic.

10. Show that $\mathrm{Aut}(C_n) \cong \mathbb{Z}_n^*$.

11. The set of all *inner automorphisms* $\mathrm{Inn}(G)$ of a group G is the set of all automorphisms $\alpha_g : G \to G$ of the form:
$$\alpha_g(x) = gxg^{-1};$$
that is, it is the set of all automorphisms given by conjugation.

    (a) Show that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

    (b) Show that:
    $$\mathrm{Inn}(G) \cong \frac{G}{Z(G)},$$
    where $Z(G)$ is the centre of the group G, defined in Examples Sheet 5, Question 4.

    (c) Identify standard groups to which $\mathrm{Inn}(C_n)$, $\mathrm{Inn}(D_2)$ and $\mathrm{Inn}(D_3)$ are isomorphic.

    The set of all *outer automorphisms* is defined to be $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$.

    (d) Identify standard groups to which $\mathrm{Out}(C_n)$, $\mathrm{Out}(D_2)$ and $\mathrm{Out}(D_3)$ are isomorphic.

12. Show that $\mathrm{Aut}(Q_8) \cong O$, where O is the chiral cubic group. Identify standard groups to which $\mathrm{Inn}(Q_8)$ and $\mathrm{Out}(Q_8)$ are isomorphic.