Clarkson Institute Summer School: Examples Sheet 7 - Permutation groups

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Symmetric and alternating groups

1. Calculate the inverse of the permutation $(123)(45) \in S_5$.
2. Decompose the following permutations into: (i) their disjoint cycles; (ii) either an even or odd number of transpositions, as appropriate.
   (a) $(12)(1234)(12)$;
   (b) $(123)(1234)(132)$;
   (c) $(123)(235)(345)(45)$.
3. Suppose that a permutation has a disjoint cycle representation comprising cycles of lengths $n_1, n_2, ..., n_k$. Show that the order of the permutation is the least common multiple of $n_1, n_2, ..., n_k$. Hence, find the orders of the permutations in Question 1.
4. Let $\sigma : \mathbb{Z}_{31} \to \mathbb{Z}_{31}$ be defined by $\sigma(x) = 2x$. Show that σ is a permutation. Reduce σ to its disjoint cycle representation, and find its order.
5. Find the largest possible orders of elements in S_5 and A_5, respectively.
6. Show that the symmetric group S_n is generated by each of the following subsets:
   (a) $\{(j, j+1) \,:\, 1 \leq j < n\}$;
   (b) $\{(1, k) \,:\, 1 < k \leq n\}$;
   (c) $\{(12), (12....n)\}$.

Cayley's theorem

7. Prove the following generalisation of Cayley's theorem: if $H \leq G$ is a subgroup of a group G, then there exists a normal subgroup $K \trianglelefteq G$ with $K \leq H$ such that G/K is isomorphic to a subgroup of $\mathrm{Sym}(G/H)$. [*Here, G/H denotes the set of left cosets of H in G, which is not a group unless H is normal. Note that Cayley's theorem is the special case $H = \{e\}$.*]
8. Show that every finite group is isomorphic to a subgroup of an alternating group.
9. Prove that S_n has a subgroup isomorphic to Q_8 if and only if $n \geq 8$. [*Hint: consider $H \subseteq Q_8$, the set of all elements whose image permutations fix 1. Begin by proving this is a subgroup, and consider its cosets.*]

Conjugacy in permutation groups

10. By considering the possible cycle types, determine the sizes of all the conjugacy classes in both S_4 and S_5.
11. (a) By considering the chiral cubic group O, give a geometrical interpretation of the conjugacy classes in S_4.
    (b) By considering the chiral tetrahedral group T, give a geometrical interpretation of the conjugacy classes in A_4.
12. Show that A_5 is not a simple group. [*Hint: start by calculating the sizes of the conjugacy classes in A_5.*]