Clarkson Institute Summer School:
Examples Sheet 8 - Permutation representations and linear representations

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Permutation representations

1. Consider the permutation representation $f : D_4 \to \mathrm{Sym}(\{1,2,3,4\})$ where $\{1,2,3,4\}$ are the vertices of a square. Identify the orbit and stabiliser of 1, and check that the orbit-stabiliser theorem is valid.

2. We define the *conjugation action* to be the map $f : G \to \mathrm{Sym}(G)$ given by $f(g) = \alpha_g$, where $\alpha_g(h) = ghg^{-1}$. Show that this indeed gives a permutation representation. Identify the orbit and stabiliser of a given element $x \in G$ with an object from earlier in the course. What does the orbit-stabiliser theorem tells you in this case?

3. Suppose that G is a finite simple group of order greater than two, and suppose that $H \leq G$ is a proper subgroup with n cosets in G. Show that $|G| \leq n!$.

Definitions and examples of linear representations

4. In each of the following cases, decide whether or not the following assignments of matrices to group generators extend to representations. Justify your answers.

   (a) $D : \mathbb{Z} \to GL(2, \mathbb{C})$, given by $D(1) = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$;

   (b) $D : \mathbb{Z} \to GL(2, \mathbb{C})$, given by $D(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$;

   (c) $D : \mathbb{Z}_4 \to GL(2, \mathbb{C})$, given by $D(1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$;

   (d) $D : S_3 \to GL(2, \mathbb{C})$, given by $D((12)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $D((23)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

   Of those that are representations, which are faithful?

5. Find all one-dimensional representations of: (a) D_n; (b) Q_8.

6. Construct the left regular representations of C_2 and C_3.

7. Suppose that $D : G \to GL(n, \mathbb{C})$ is a representation. Show that $\overline{D} : G/\mathrm{Ker}(f) \to GL(n, \mathbb{C})$ given by $\overline{D}(g\mathrm{Ker}(f)) = D(g)$ is a faithful representation of $G/\mathrm{Ker}(f)$.

Equivalence and inequivalence of linear representations

8. Show that the following maps extend to representations of $C_3 = \{e, r, r^2\}$:

$$D_1(r) = \begin{pmatrix} e^{2\pi i/3} & 0 \\ 0 & e^{4\pi i/3} \end{pmatrix}, \qquad D_2(r) = \begin{pmatrix} e^{4\pi i/3} & 0 \\ 0 & e^{2\pi i/3} \end{pmatrix}.$$

   Show that they are equivalent by explicitly identifying an intertwining matrix.

9. Show that the following maps extend to representations of $C_4 = \{e, r, r^2, r^3\}$:

$$D_1(r) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad D_2(r) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Show that the representations are equivalent, and that the general intertwining matrix between them is given by:

$$S = \begin{pmatrix} a & b \\ ia & -ib \end{pmatrix},$$

where a, b are non-zero complex numbers.

10. Prove that for any 2×2 complex matrices A, B, we have $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. Can you extend your proof to $n \times n$ complex matrices? [*Hint: if $C = AB$ is the product of two matrices A, B, how can we express C_{ij} (the (i, j)th entry of C) in terms of the entries of A, B?*]

11. For the maps in Question 4 which are representations, evaluate their characters χ_D in each case.

12. Show that the following map extends to a representation of $C_4 = \{e, r, r^2, r^3\}$:

$$D_3(r) = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}.$$

Is it equivalent to the representations in Question 9?

13. Let $D : G \to GL(n, \mathbb{C})$ be a representation of the group G. Prove that:

    (a) $\chi_D(e) = n$;

    (b) if x, y lie in the same conjugacy class in G, then $\chi_D(x) = \chi_D(y)$.