Clarkson Institute Summer School: Examples Sheet 9 - Reducibility to direct sums

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Complex vector spaces

1. Calculate the standard complex inner product of the vectors $(1, i, 1)$ and $(i, -1, 0)$ in $\mathbb{C}^3$.

2. Show that:
$$(\mathbf{v}, \mathbf{w}) = \begin{pmatrix} v_1^* & v_2^* & v_3^* \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$
defines an abstract inner product on $\mathbb{C}^3$. Evaluate the inner product of the vectors in Question 1 with respect to this inner product.

3. Apply the Gram-Schmidt procedure to produce an orthogonal basis from $\{(1,0,0),(1,1,0),(1,1,1)\}$ for $\mathbb{C}^3$ with respect to the abstract inner product defined in Question 2.

4. Show that the matrix:
$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix}$$
is unitary.

Reducibility to direct sums

5. (a) Consider the left regular representation D of C_3, comprising of three 3×3 matrices. Identify an invariant one-dimensional subspace and an invariant two-dimensional subspace of this representation.
   (b) Extend the idea of your proof in part (a) to show that the left regular representation of any non-trivial finite group is always reducible.

6. Show that for $n \geq 3$, the representation D of D_n generated by:
$$D(r) = \begin{pmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{pmatrix}, \qquad D(m) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
is irreducible.

7. Let $D : G \to GL(2, \mathbb{C})$ be a two-dimensional reducible representation. Show that there exists a non-zero vector $\mathbf{v} \in \mathbb{C}^2$ such that:
$$D(g)\mathbf{v} = \lambda(g)\mathbf{v},$$
where $\lambda : G \to \mathbb{C}^*$ is a one-dimensional representation of G.

8. Let $D : G \to GL(n, \mathbb{C})$ be an irreducible representation, and let $d : G \to \mathbb{C}^*$ be a one-dimensional representation. Show that $D' : G \to GL(n, \mathbb{C})$ defined by $D'(g) = d(g)D(g)$ is irreducible.

9. Let D_1, D_2 be representations. Show that $D_1 \oplus D_2$ is equivalent to $D_2 \oplus D_1$.

Unitarity and Maschke's theorem

10. Consider the two-dimensional representation D of D_2 generated by:

$$D(r) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad D(m) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Identify two invariant subspaces of the representation. Show that it is completely reducible to a direct sum of one-dimensional representations, which you should identify.

11. Consider the two-dimensional representation D of C_n generated by:

$$D(r) = \begin{pmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{pmatrix}.$$

(a) Show that this representation is unitary with respect to the standard complex scalar product.

(b) Show that there are two invariant subspaces V_+, V_- with $\{(1,i)\}$ a basis for V_+ and $\{(1,-i)\}$ a basis for V_-. Show that these are mutually orthogonal subspaces with respect to the standard complex scalar product on $\mathbb{C}^2$.

(c) Find an intertwining matrix S such that $S^{-1}D(r)S$ is block diagonal. Hence, show that $D \cong D_+ \oplus D_-$, where $D_+(r) = e^{2\pi i/n}$ and $D_-(r) = e^{-2\pi i/n}$ are one-dimensional irreducible representations.

12. Consider the two-dimensional representation D of C_2 generated by:

$$D(r) = \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix}.$$

(a) Show that this indeed extends to a representation of C_2.

(b) Show that D is not unitary with respect to the standard complex scalar product.

(c) Construct a new abstract complex scalar product $(\cdot,\cdot)$, as per the proof of Maschke's theorem, and show that it is given by:

$$(\mathbf{v}, \mathbf{w}) = \begin{pmatrix} v_1^* & v_2^* \end{pmatrix} \begin{pmatrix} 2 & -2 \\ -2 & 6 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

(d) Show that V_+, V_- are invariant subspaces, where V_+ has basis $\{(1,0)\}$ and V_- has basis $\{(1,1)\}$. Verify that these are mutually orthogonal with respect to the abstract complex scalar product defined above.

(e) Construct an intertwining matrix that transforms D to block diagonal form. Hence decompose D into a direct sum of irreducible one-dimensional representations.