Lie algebras from Lie groups

Abstract

We review the construction of the Lie algebra of a Lie group, in particular producing a *Lie bracket* for the algebra. We begin with a quick recap of the basic definitions and notation that will be needed; we assume familiarity with basic differential geometry, such as the notion of a smooth manifold, tangent spaces, and vector fields on a smooth manifold.

Next, we introduce the Lie algebra of a Lie group as the tangent space to the identity of the Lie group. There is a natural isomorphism between this tangent space and the vector space of *left-invariant vector fields* on the Lie group, which allows us to endow the Lie algebra with a Lie bracket structure.

Finally, we describe the abstract construction in the language of local coordinates, and derive an expression for the structure constants of a Lie algebra in a coordinate basis.

1: Recap of definitions and notation

We begin by recalling the definition of a Lie group from lectures:

Definition: A *Lie group* is a smooth manifold G which is also a group with respect to some multiplication $*$, such that:

(i) Inversion in the group is a smooth map between manifolds, $(-)^{-1} : G \to G$.

(ii) Multiplication in the group is a smooth map between manifolds, $* : G \times G \to G$, where $G \times G$ is endowed with the standard product manifold structure.

Some important definitions and notation from differential geometry[1] associated with Lie groups are the following:

- The space of all smooth functions $f : G \to \mathbb{R}$ will be written $C^\infty(G)$.
- At each point $g \in G$ of the Lie group, we may construct a vector space T_gG called the *tangent space* to the group at the point $g \in G$. In these notes, we shall define T_gG to be the set of all *derivations at the point* g, i.e. functions $X : C^\infty(G) \to \mathbb{R}$ which satisfy:

  (i) <u>Linearity.</u> For all $f_1, f_2 \in C^\infty(G)$ and $a, b \in \mathbb{R}$, we have $X(af_1 + bf_2) = aX(f_1) + bX(f_2)$.

  (ii) <u>The Leibniz rule.</u> For all $f_1, f_2 \in C^\infty(G)$, we have $X(f_1 \cdot f_2) = X(f_1)f_2(p) + f_1(p)X(f_2)$, where $\cdot$ denotes point-wise multiplication.

  We naturally think of derivations at the point $g \in G$ as differential operators acting on a space of functions; indeed, it can be shown that the space T_gG is spanned by the derivatives with respect to some system of local coordinates.[2]

- A *vector field* on the Lie group G is a *global* version of a derivation,[3] in that it is a function $\mathbf{v} : C^\infty(G) \to C^\infty(G)$ satisfying:

  (i) <u>Linearity.</u> For all $f_1, f_2 \in C^\infty(G)$ and $a, b \in \mathbb{R}$, we have $\mathbf{v}(af_1 + bf_2) = a\mathbf{v}(f_1) + b\mathbf{v}(f_2)$.

  (ii) <u>The Leibniz rule.</u> For all $f_1, f_2 \in C^\infty(G)$, we have $\mathbf{v}(f_1 \cdot f_2) = \mathbf{v}(f_1) \cdot f_2 + f_1 \cdot \mathbf{v}(f_2)$.

  The set of all vector fields on G will be written as $\mathrm{Vect}(G)$ throughout. Note that for each $g \in G$, there is a natural projection $\mathrm{pr}_g : \mathrm{Vect}(G) \to T_gG$ given by $\mathrm{pr}_g(\mathbf{v})(f) := \mathbf{v}(f)(g)$ for all $f \in C^\infty(G)$; throughout, we write the image of $\mathbf{v}$ under pr_g as $\mathbf{v}_g$.

[1] For more information, see Part III Differential Geometry or Part III General Relativity.

[2] See Harvey Reall's Part III General Relativity notes for a proof.

[3] This is one possible definition of a vector field; equivalently, we may define a vector field to be a smooth section of the tangent bundle of the Lie group, but this requires some technology which is often not developed in physics courses.

It will also be useful to recall that the space of all vector fields has a natural vector space structure, together with a natural *commutator*.

Proposition: We have the following:

(i) The space of all vector fields $\mathrm{Vect}(G)$ is naturally a (real) vector space.

(ii) The *commutator* of vector fields, namely:

$$[\mathbf{v}, \mathbf{w}] := \mathbf{v} \circ \mathbf{w} - \mathbf{w} \circ \mathbf{v},$$

is a vector field.

Proof: (i) is obvious if we define $(a\mathbf{v} + b\mathbf{w})(f) := a\mathbf{v}(f) + b\mathbf{w}(f)$ for all $a, b \in \mathbb{R}$, $\mathbf{v}, \mathbf{w} \in \mathrm{Vect}(G)$ and $f \in C^\infty(G)$. (ii) requires checking linearity and the Leibniz rule, and is left as an easy exercise to the reader. $\square$

2: Abstract construction of the Lie algebra

After the recap of definitions and notation from differential geometry, we are ready to introduce Lie algebras and endow them with a bracket structure.

Definition: The *Lie algebra* $\mathfrak{g}$ of the Lie group G is the tangent space to G at the identity $e \in G$, i.e. $\mathfrak{g} := T_e G$.

Currently, the Lie algebra $\mathfrak{g}$ only has the structure of a vector space. It has no natural commutator structure - it is meaningless to ask for the commutator $[X, Y]$ for $X, Y \in \mathfrak{g}$, since X, Y are both *derivations at the point* $e \in G$. In particular, a composition like $X \circ Y$ makes no sense, since both X, Y take in functions from $C^\infty(G)$ and return real numbers.

On the other hand, if we could somehow extend the derivations X, Y at the identity to vector fields in a natural way, we would be able to take the commutator. This can be achieved naturally using the group multiplication structure:

Notation: Let $\mathfrak{g}$ be the Lie algebra of the Lie group G. For each derivation $X \in \mathfrak{g}$ at the identity, we define an associated vector field $\theta(X) : C^\infty(G) \to C^\infty(G)$ via:

$$\theta(X)_g(f) := X(f \circ L_g),$$

for all $f \in C^\infty(G)$ and $g \in G$. Here, $L_g : G \to G$ denotes left-multiplication by g, i.e. $L_g(h) = gh$. As an exercise, you should check that $\theta(X)_g$ is indeed a derivation at the point $g \in G$, so that this vector field is well-defined (indeed, technically one must also check that the image function it defines is also smooth; this is the case).

This has the following natural interpretation. The derivation $X \in \mathfrak{g}$ acts as a differential operator at the point $e \in G$. In order to construct an associated vector field, i.e. construct an associated set of derivations at every point in the group, we must 'transport' this differential operator to every other point in the group. To do so, we define a differential operator at the point $g \in G$ by first left-multiplying any argument of the function f by g (since the value of $f \circ L_g$ at e is the same as the value of f at g), then we apply the derivation $X \in \mathfrak{g}$.

Note that it is not possible to perform this construction if G is not a Lie group, just a smooth manifold; it was *essential* to have the group multiplication structure available to us.

We are now a bit closer to making sense of the 'commutator' of $X, Y \in \mathfrak{g}$. We would like to say:

$$[X, Y] := \theta^{-1}([\theta(X), \theta(Y)]).$$

In particular, the function $\theta : \mathfrak{g} \to \mathrm{Vect}(G)$ which extends derivations at $e \in G$ to vector fields must necessarily be a *bijection*. Is this the case?

Unfortunately, the answer is ***no***. It turns out that all vector fields $\theta(X)$ for $X \in \mathfrak{g}$ have a special property called *left-invariance* which is not possessed by general vector fields. Restricting the codomain of θ to left-invariant vector fields fixes our problem:

Definition: A vector field $\mathbf{v} : C^\infty(G) \to C^\infty(G)$ is called *left-invariant* if for all $g, h \in G$ and $f \in C^\infty(G)$, we have:

$$\mathbf{v}_{L_g(h)}(f) = \mathbf{v}_h(f \circ L_g).$$

We write the set of all left-invariant vector fields as $\mathrm{Vect}_L(G)$. As an exercise, one can check that $\mathrm{Vect}_L(G)$ is a vector subspace of $\mathrm{Vect}(G)$. Furthermore, if $\mathbf{v}, \mathbf{w}$ are left-invariant, one can show that $[\mathbf{v}, \mathbf{w}]$ is additionally left-invariant, so that $\mathrm{Vect}_L(G)$ is closed under the commutator; this is an easy exercise left to the reader.

Proposition: The map $\theta : \mathfrak{g} \to \mathrm{Vect}_L(G)$ is a vector space isomorphism.

Proof: First, note that the map is well-defined (i.e. $\theta(X)$ is indeed left-invariant for all $X \in \mathfrak{g}$). We can prove this simply as follows; note for all $g, h \in G$ and $f \in C^\infty(G)$ we have:

$$\begin{aligned}
\theta(X)_{L_g(h)}(f) &= \theta(X)_{gh}(f) \\
&= X(f \circ L_{gh}) \\
&= X((f \circ L_g) \circ L_h) \\
&= \theta(X)_h(f \circ L_g).
\end{aligned}$$

Next, note that the map is linear; for all $a, b \in \mathbb{R}$, $X, Y \in \mathfrak{g}$ and $f \in C^\infty(G)$, we have:

$$\begin{aligned}
\theta(aX + bY)_g(f) &= (aX + bY)(f \circ L_g) \\
&= aX(f \circ L_g) + bY(f \circ L_g) \\
&= a\theta(X)_g(f) + b\theta(Y)_g(f),
\end{aligned}$$

Finally, we must show that the map is a bijection. We do so by producing an inverse map, which we define by $\theta^{-1}(\mathbf{v}) = \mathbf{v}_e$. This is clearly well-defined, and provides an inverse to θ since:

$$\theta^{-1}(\theta(X))(f) = \theta(X)_e(f) = X(f)$$

for all $f \in C^\infty(G)$ and all $X \in \mathfrak{g}$, and:

$$\theta(\theta^{-1}(\mathbf{v}))_g(f) = \theta(\mathbf{v}_e)_g(f) = \mathbf{v}_e(f \circ L_g) = \mathbf{v}_g(f),$$

for all $g \in G$, $f \in C^\infty(G)$ and $\mathbf{v} \in \mathrm{Vect}_L(G)$ (the last equality follows from left-invariance). $\square$

After all that work, we are now ready to finish our construction of the Lie algebra, by defining an appropriate commutator:

Definition: We define the *Lie bracket* of $X, Y \in \mathfrak{g}$ to be:

$$[X, Y] = \theta^{-1}([\theta(X), \theta(Y)]),$$

where $\theta : \mathfrak{g} \to \mathrm{Vect}_L(G)$ is the vector space isomorphism defined above.

One can check that this commutator is linear in both arguments, is antisymmetric, and obeys the Jacobi identity; these are the standard properties of a Lie bracket.

3: The Lie algebra in local coordinates

The above work can be translated entirely into local coordinates. Let G be an n-dimensional Lie group, and suppose that (U, ϕ) is a coordinate chart centred on $e \in G$ (i.e. ϕ is a homeomorphism $\phi : U \to \phi(U) \subseteq \mathbb{R}^n$ satisfying $\phi(e) = \mathbf{0}$). Write $\phi = (x^1, ..., x^n)$ for the individual components of ϕ. Given any element of the coordinate patch, we can label it by its coordinates; let us write $g(\mathbf{x})$ for the unique element in U obeying $\phi(g(\mathbf{x})) = \mathbf{x}$.

For group elements with sufficiently small coordinates, we can describe their multiplication via a smooth function:

Proposition: There exists an open subset $V \subseteq U$ such that for all $g(\mathbf{x}), g(\mathbf{y}) \in V$, we have $g(\mathbf{x})g(\mathbf{y}) \in U$. Furthermore, we may write:

$$g(\mathbf{x})g(\mathbf{y}) = g(\mu(\mathbf{x}, \mathbf{y}))$$

for some smooth function $\mu : \phi(V) \times \phi(V) \to \phi(U)$.

Proof: We use the smoothness of the multiplication of the Lie group. Recall that $* : G \times G \to G$ is a smooth map between manifolds, so in particular is a continuous map between topological spaces. Since U is an open neighbourhood of the identity, it follows that $*^{-1}(U)$ is an open neighbourhood of (e, e) in $G \times G$, since $e * e = e$. By definition of the product topology, it follows that we can write:

$$*^{-1}(U) = \bigcup_\alpha U^1_\alpha \times U^2_\alpha,$$

for some open subsets $U^1_\alpha, U^2_\alpha \subseteq G$ with at least one α^* such that $(e, e) \in U^1_{\alpha^*} \times U^2_{\alpha^*}$. Fix such an α^*, and define $V = U^1_{\alpha^*} \cap U^2_{\alpha^*}$. Then V is open since it is the finite intersection of open sets, and $e \in V$ since $(e, e) \in U^1_{\alpha^*} \times U^2_{\alpha^*}$. Note also that $U^1_{\alpha^*} \subseteq U$ since $U^1_{\alpha^*} = *(U^1_{\alpha^*} \times \{e\}) \subseteq U$, hence $V \subseteq U$. Finally, note $V \times V \subseteq (\cdot)^{-1}(U)$, so V has the required multiplication property.

We have now shown that there exists $V \subseteq U$ such that for all $g(\mathbf{x}), g(\mathbf{y}) \in V$, we have $g(\mathbf{x})g(\mathbf{y}) \in U$. Thus we can write $g(\mathbf{x})g(\mathbf{y}) = g(\mu(\mathbf{x}, \mathbf{y}))$ for some function $\mu : \phi(V) \times \phi(V) \to \phi(U)$. It remains to show that μ is smooth in the sense of real, multivariable calculus; this follows almost immediately from the smoothness of the group multiplication $* : G \times G \to G$ in the manifold sense. We note that (U, ϕ) is a chart on G, and $(V \times V, \phi|_V \times \phi|_V)$ is a chart on $G \times G$. Therefore, it follows that:

$$\mu = \phi \circ * \circ (\phi|_V \times \phi|_V)^{-1} : \phi(V) \times \phi(V) \to \phi(U).$$

is smooth, as required. $\square$

To construct the Lie algebra in terms of local coordinates, we recall[4] that the Lie algebra $\mathfrak{g}$, i.e. the tangent space at the identity, can be spanned by a collection of differential operators:

$$\left\{ \frac{\partial}{\partial x^i}\bigg|_e := \frac{\partial(- \circ \phi^{-1})}{\partial x^i}\bigg|_{\mathbf{x}=\mathbf{0}} : i = 1, ..., n \right\}.$$

This choice of basis is called a *coordinate basis* for the Lie algebra $\mathfrak{g}$. Note that the action of the coordinate basis operators on smooth functions $f : G \to \mathbb{R}$ can be neatly written in terms of the labelled group elements $g(\mathbf{x}) \in U$; we have:

$$\frac{\partial}{\partial x^i}\bigg|_e (f) = \frac{\partial(f \circ \phi^{-1})}{\partial x^i}\bigg|_{\mathbf{x}=\mathbf{0}} = \frac{\partial(f(g(\mathbf{x})))}{\partial x^i}\bigg|_{\mathbf{x}=\mathbf{0}},$$

using the fact that $g(\mathbf{x})$ is the unique element of U satisfying $\phi(g(\mathbf{x})) = \mathbf{x}$.

It is interesting to ask what the Lie bracket of two basis elements looks like in local coordinates. In order to discover this, we need to first construct the left-invariant vector fields associated with each of the basis elements. Let $g(\mathbf{y}) \in V$. Then we have:

$$\begin{aligned}
\theta\left(\frac{\partial}{\partial x^i}\bigg|_e\right)_{g(\mathbf{y})} (f) &:= \frac{\partial}{\partial x^i}\bigg|_e (f \circ L_{g(\mathbf{y})}) \\
&= \frac{\partial(f \circ L_{g(\mathbf{y})} \circ \phi^{-1})}{\partial x^i}\bigg|_{\mathbf{x}=\mathbf{0}} \\
&= \frac{\partial(f(g(\mathbf{y})g(\mathbf{x})))}{\partial x^i}\bigg|_{\mathbf{x}=\mathbf{0}} \\
&= \frac{\partial(f \circ \phi^{-1}(\mu(\mathbf{y}, \mathbf{x})))}{\partial x^i}\bigg|_{\mathbf{x}=\mathbf{0}}.
\end{aligned}$$

We can apply the chain rule to the right hand side to simplify it. We obtain (with the summation convention implied):

$$\theta\left(\frac{\partial}{\partial x^i}\bigg|_e\right)_{g(\mathbf{y})} (f) = \frac{\partial \mu^r(\mathbf{y}, \mathbf{x})}{\partial x^i}\bigg|_{\mathbf{x}=\mathbf{0}} \frac{\partial(f \circ \phi^{-1})}{\partial x^r}\bigg|_{\mathbf{x}=\mathbf{y}} = \frac{\partial \mu^r(\mathbf{y}, \mathbf{x})}{\partial x^i}\bigg|_{\mathbf{x}=\mathbf{0}} \frac{\partial(f(g(\mathbf{y})))}{\partial y^r},$$

which can be written succinctly as:

$$\theta\left(\frac{\partial}{\partial x^i}\bigg|_e\right)_{g(\mathbf{y})} (f) = \mu^r{}_i(\mathbf{y}) \frac{\partial(f(g(\mathbf{y})))}{\partial y^r},$$

where we have defined $\mu^r{}_i(\mathbf{y}) = \partial \mu^r(\mathbf{y}, \mathbf{x})/\partial x^i|_{\mathbf{x}=\mathbf{0}}$. Thus we have obtained local coordinate expressions for the left-invariant vector fields. In an abuse of notation, we could further write:

$$\mu^r{}_i(\mathbf{y}) \frac{\partial(f(g(\mathbf{y})))}{\partial y^r} = \mu^r{}_i(\mathbf{y}) \frac{\partial}{\partial y^r}\bigg|_{g(\mathbf{y})} (f),$$

where it is understood that we should evaluate the function at $g(\mathbf{y})$, then take the derivative with respect to y^r. We summarise this below (and perform some relabelling, $\mathbf{y} \mapsto \mathbf{x}$).

[4] For example, from Part III General Relativity.

Notation: We write the left-invariant vector field associated to the ith coordinate basis vector as T_i. It is given locally by:

$$T_{i,g(\mathbf{x})} := \mu^r{}_i(\mathbf{x}) \frac{\partial}{\partial x^r}\bigg|_{g(\mathbf{x})}.$$

The coefficients $\mu^r{}_i(\mathbf{x})$ are given in terms of the local multiplication function $\mu : \phi(V) \times \phi(V) \to \phi(U)$ of the Lie group via:

$$\mu^r{}_i(\mathbf{x}) := \frac{\partial \mu^r(\mathbf{x}, \boldsymbol{\theta})}{\partial \theta^i}\bigg|_{\boldsymbol{\theta}=\mathbf{0}}.$$

With the appropriate restriction, μ can be considered a smooth *bijection* $\mu(\mathbf{x}, -)$ for each fixed $\mathbf{x}$. In this case, $\mu^r{}_i(\mathbf{x})$ is simply the Jacobian matrix of this bijection at the point $\mathbf{x}$, so is in particular an *invertible* matrix for each $\mathbf{x}$. We write the inverse of $\mu^r{}_i(\mathbf{x})$ as $\lambda^i{}_s(\mathbf{x})$, such that:

$$\mu^r{}_i(\mathbf{x})\lambda^i{}_s(\mathbf{x}) = \delta^r{}_s.$$

In this language, there is a nice translation of the left-invariance property of the vector fields T_i:

Proposition: For all $\mathbf{x}, \mathbf{y} \in \phi(V)$, we have:

$$\mu^s{}_i(\mu(\mathbf{x}, \mathbf{y})) = \mu^r{}_i(\mathbf{y}) \frac{\partial \mu^s(\mathbf{x}, \mathbf{y})}{\partial y^r}.$$

Proof: Let $g(\mathbf{x}), g(\mathbf{y}) \in V$ and let $g(\mathbf{z}) = g(\mathbf{x})g(\mathbf{y}) \in U$, so that $\mathbf{z} = \mu(\mathbf{x}, \mathbf{y})$. We now use the left-invariance of the vector field T_i. We have:

$$T_{i,g(\mathbf{x})g(\mathbf{y})}(f) = T_{i,g(\mathbf{z})}(f) = \mu^s{}_i(\mathbf{z}) \frac{\partial}{\partial z^s}\bigg|_{g(\mathbf{z})}(f).$$

By left-invariance, this is also equal to:

$$\begin{aligned} T_{i,g(\mathbf{y})}(f \circ L_{g(\mathbf{x})}) &= \mu^r{}_i(\mathbf{y}) \frac{\partial}{\partial y^r}\bigg|_{g(\mathbf{y})}(f \circ L_{g(\mathbf{x})}) = \mu^r{}_i(\mathbf{y}) \frac{\partial\left(f(g(\mathbf{x})g(\mathbf{y}))\right)}{\partial y^r} \\ &= \mu^r{}_i(\mathbf{y}) \frac{\partial\left(f(g(\mathbf{z}))\right)}{\partial y^r} = \mu^r{}_i(\mathbf{y}) \frac{\partial z^s}{\partial y^r} \frac{\partial}{\partial z^s}\bigg|_{g(\mathbf{z})}(f). \end{aligned}$$

Comparing coefficients, the result follows. □

We can now construct the Lie bracket in local coordinates. Recall that by definition, we have:

$$\left[\frac{\partial}{\partial x^i}\bigg|_e, \frac{\partial}{\partial x^j}\bigg|_e\right] = \theta^{-1}\left([T_i, T_j]\right),$$

so in order to evaluate the Lie bracket in local coordinates, we need to evaluate the commutator $[T_i, T_j]$ of left-invariant vector fields. Consider evaluating the projection of the commutator onto a specific point, $[T_i, T_j]_{g(\mathbf{x})}$. For any smooth function $f \in C^\infty(G)$, this is given by:[5]

$$\begin{aligned} [T_i, T_j]_{g(\mathbf{x})}(f) &= T_{i,g(\mathbf{x})} \circ T_j(f) - T_{j,g(\mathbf{x})} \circ T_i(f) \\ &= T_{i,g(\mathbf{x})}\left(g(\mathbf{y}) \mapsto \mu^s{}_j(\mathbf{y}) \frac{\partial}{\partial y^s}\bigg|_{g(\mathbf{y})}(f)\right) - T_{j,g(\mathbf{x})}\left(g(\mathbf{y}) \mapsto \mu^s{}_i(\mathbf{y}) \frac{\partial}{\partial y^s}\bigg|_{g(\mathbf{y})}(f)\right) \end{aligned}$$

[5] Some thought is required in each step of this derivation to keep track of what is a function, and what is a number.

$$= \mu^r{}_i(\mathbf{x}) \frac{\partial}{\partial x^r} \left(\mu^s{}_j(\mathbf{x}) \frac{\partial}{\partial x^s}(f(g(\mathbf{x}))) \right) - \mu^r{}_j(\mathbf{x}) \frac{\partial}{\partial x^r} \left(\mu^s{}_i(\mathbf{x}) \frac{\partial}{\partial x^s}(f(g(\mathbf{x}))) \right)$$

$$= \left(\mu^r{}_i(\mathbf{x}) \frac{\partial \mu^s{}_j(\mathbf{x})}{\partial x^r} - \mu^r{}_j(\mathbf{x}) \frac{\partial \mu^s{}_i(\mathbf{x})}{\partial x^r} \right) \frac{\partial}{\partial x^s}(f(g(\mathbf{x})))$$

$$= \left(\mu^r{}_i(\mathbf{x}) \frac{\partial \mu^s{}_j(\mathbf{x})}{\partial x^r} - \mu^r{}_j(\mathbf{x}) \frac{\partial \mu^s{}_i(\mathbf{x})}{\partial x^r} \right) \lambda^k{}_s(\mathbf{x}) \mu^a{}_k(\mathbf{x}) \left. \frac{\partial}{\partial x^a} \right|_{g(\mathbf{x})} (f)$$

$$= \left(\mu^r{}_i(\mathbf{x}) \frac{\partial \mu^s{}_j(\mathbf{x})}{\partial x^r} - \mu^r{}_j(\mathbf{x}) \frac{\partial \mu^s{}_i(\mathbf{x})}{\partial x^r} \right) \lambda^k{}_s(\mathbf{x}) T_{k,g(\mathbf{x})}(f).$$

The coefficient on the right hand side of the left-invariant vector field is called a *structure constant* for the Lie algebra:

Definition: With respect to the local coordinate basis $\partial/\partial x^i|_e$, the Lie algebra has *structure constants*:

$$f^k{}_{ij}(\mathbf{x}) := \left(\mu^r{}_i(\mathbf{x}) \frac{\partial \mu^s{}_j(\mathbf{x})}{\partial x^r} - \mu^r{}_j(\mathbf{x}) \frac{\partial \mu^s{}_i(\mathbf{x})}{\partial x^r} \right) \lambda^k{}_s(\mathbf{x}).$$

With this definition, the commutator of the left-invariant vector fields simply becomes:

$$[T_i, T_j]_{g(\mathbf{x})}(f) = f^k{}_{ij}(\mathbf{x}) T_{k,g(\mathbf{x})}(f).$$

It remains to translate this back to the Lie algebra, i.e. apply the inverse map θ^{-1}. This can be done immediately and simply, provided we first show that the coefficient $f^k{}_{ij}(\mathbf{x})$ is in fact a real *constant*, i.e. the $\mathbf{x}$-dependence is superficial. This can be achieved straightforwardly using the condition for left-invariance we derived above:

Proposition: For any $\mathbf{x}, \mathbf{y} \in \phi(V)$, we have $f^k{}_{ij}(\mathbf{x}) = f^k{}_{ij}(\mathbf{y})$.

Proof: Note first that there exists an element $g(\mathbf{z}) \in G$ such that $g(\mathbf{x}) = g(\mathbf{z})g(\mathbf{y})$ (namely $g(\mathbf{z}) = g(\mathbf{x})g(\mathbf{y})^{-1}$). By smoothness of multiplication and inversion, it can be shown (by a similar argument to the above) that $g(\mathbf{z}) \in V$ without loss of generality. In particular, there exists a unique $\mathbf{z}$ such that $\mathbf{x} = \mu(\mathbf{z}, \mathbf{y})$. Now, by the condition for left-invariance from above, we have:

$$\mu^s{}_i(\mathbf{x}) = \mu^r{}_i(\mathbf{y}) \frac{\partial x^s}{\partial y^r}. \tag{$*$}$$

Inverting some matrices, this also implies the relation:

$$\lambda^k{}_s(\mathbf{x}) \frac{\partial x^s}{\partial y^r} = \lambda^k{}_r(\mathbf{y}). \tag{$\dagger$}$$

Substituting using $(*)$, and then $(\dagger)$ on the penultimate line, we have:

$$f^k{}_{ij}(\mathbf{x}) = \left(\mu^r{}_i(\mathbf{x}) \frac{\partial \mu^s{}_j(\mathbf{x})}{\partial x^r} - \mu^r{}_j(\mathbf{x}) \frac{\partial \mu^s{}_i(\mathbf{x})}{\partial x^r} \right) \lambda^k{}_s(\mathbf{x})$$

$$= \left(\mu^a{}_i(\mathbf{y}) \frac{\partial x^r}{\partial y^a} \frac{\partial}{\partial x^r} \left(\mu^b{}_j(\mathbf{y}) \frac{\partial x^s}{\partial y^b} \right) - \mu^a{}_j(\mathbf{y}) \frac{\partial x^r}{\partial y^a} \frac{\partial}{\partial x^r} \left(\mu^b{}_i(\mathbf{y}) \frac{\partial x^s}{\partial y^b} \right) \right) \lambda^k{}_s(\mathbf{x})$$

$$
\begin{aligned}
&= \left(\mu^a{}_i(\mathbf{y}) \frac{\partial}{\partial y^a}\left(\mu^b{}_j(\mathbf{y}) \frac{\partial x^s}{\partial y^b}\right) - \mu^a{}_j(\mathbf{y}) \frac{\partial}{\partial y^a}\left(\mu^b{}_i(\mathbf{y}) \frac{\partial x^s}{\partial y^b}\right)\right) \lambda^k{}_s(\mathbf{x}) \\
&= \left(\mu^a{}_i(\mathbf{y}) \frac{\partial \mu^b{}_j(\mathbf{y})}{\partial y^a} - \mu^a{}_j(\mathbf{y}) \frac{\partial \mu^b{}_i(\mathbf{y})}{\partial y^a}\right) \frac{\partial x^s}{\partial y^b} \lambda^k{}_s(\mathbf{x}) \\
&= \left(\mu^a{}_i(\mathbf{y}) \frac{\partial \mu^b{}_j(\mathbf{y})}{\partial y^a} - \mu^a{}_j(\mathbf{y}) \frac{\partial \mu^b{}_i(\mathbf{y})}{\partial y^a}\right) \lambda^k{}_b(\mathbf{y}) \\
&= f^k{}_{ij}(\mathbf{y}).
\end{aligned}
$$

In the derivation, we used the chain rule and symmetry of mixed partial derivatives, along with the left-invariance condition from above. □

Thus, we have shown that the Lie bracket structure of $\mathfrak{g}$ is described in terms of local coordinates via:

$$
\left[\left.\frac{\partial}{\partial x^i}\right|_e, \left.\frac{\partial}{\partial x^j}\right|_e\right] = f^k{}_{ij} \left.\frac{\partial}{\partial x^k}\right|_e,
$$

where for any $\mathbf{x}$, the structure constants are given by:

$$
f^k{}_{ij} = \left(\mu^r{}_i(\mathbf{x}) \frac{\partial \mu^s{}_j(\mathbf{x})}{\partial x^r} - \mu^r{}_j(\mathbf{x}) \frac{\partial \mu^s{}_i(\mathbf{x})}{\partial x^r}\right) \lambda^k{}_s(\mathbf{x}).
$$