Part IA: Physical Natural Sciences
Preparatory Mathematics Examples Sheet

Please send all comments and corrections to jmm232@cam.ac.uk.
Starred questions, denoted ($*$), ($**$), are harder; they should not be completed at the expense of the rest of the sheet.

Topic: exponential functions

1. Find the derivatives of the following functions:

   (a) e^x, (b) $xe^{3x} - 2$, (c) e^{x^2+3}, (d) $\dfrac{e^{x^2}}{x^2+1}$, (e) 2^x, (f) x^{x^3}.

2. Sketch the graphs of the following functions:

   (a) e^x, (b) e^{2x}, (c) e^{-x}, (d) xe^{-x}, (e) $\dfrac{e^{-x}}{x}$, (f) e^{-x^2}.

3. Show that the x-coordinates of the points of intersection between the normal to the graph of $y = e^x$ at $x = 1$ and the graph of $y = e^{-x}$ satisfy:
$$x = e^2 + 1 - e^{1-x}.$$
   How many solutions do you expect to this equation? Using a numerical method of your choice, show that the unique positive solution to this equation is $x \approx 8.39$.

4. Find the indefinite integrals of the following functions:

   (a) e^x, (b) $3e^{7x} - 4$, (c) 2^x, (d) $3xe^{x^2}$, (e) $\dfrac{e^{-1/x}}{x^2}$, (f) xe^x.

5. For non-negative integers n, we define the integral I_n to be:
$$I_n := \int_0^\infty x^n e^{-x}\, dx.$$
   Using integration by parts, show that $I_n = nI_{n-1}$. Hence or otherwise, show that $I_n = n!$.

   ($*$) Hence, suggest a possible definition of $x!$ for x a *real* number. ($**$) Using your suggested definition, determine the value of $\frac{1}{2}!$. *[Hint: look up the Gaussian integral.]*

6. Using any method you know, solve the differential equation:
$$\frac{dy}{dx} = \alpha y + \beta, \qquad y(0) = y_0,$$
   where α, β are constants. What happens when $\alpha = 0$? What happens when $y_0 = -\beta/\alpha$? Taking $\alpha < 0$ and $\beta > 0$, sketch some solution curves for different values of y_0, on the same set of axes.

7. ($*$) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function satisfying:
$$f(x)f(y) = f(x+y)$$
   for all $x, y \in \mathbb{R}$. Show that $f(x) = f(1)^x$. *[Hint: take logarithms, then differentiate with respect to x, keeping y constant.]* Hence, we have shown that the property $f(x)f(y) = f(x+y)$ uniquely characterises exponential functions.

   ($**$) What if we *only* assume that f is *continuous*, instead of *differentiable* - must we have $f(x) = f(1)^x$?