Symmetries and quantum mechanics

Abstract

We discuss the implementation of symmetries in quantum mechanics via *unitary projective representations* on Hilbert spaces. We discuss the transition between projective and non-projective representations via *Bargmann's Theorem*, and end by giving applications in quantum field theory.

1: Symmetries and Wigner's theorem

Our key goal in the Symmetries, Fields and Particles course is to *implement symmetries in the quantum theory* (and see the nice things that we get when we do); hence let's quickly recap how quantum mechanics works:

Summary of Hilbert space quantum mechanics

A quantum system $\mathcal{Q}$ comprises (i) a set of physical states $\mathcal{S}$ in which it can reside, and (ii) a set of observables $\mathcal{O}$ which we can measure the system for. These qualities are realised in mathematics as follows:

- **Postulate I:** The quantum system $\mathcal{Q}$ has an associated Hilbert space $\mathcal{H}$.
- **Postulate II:** Given a non-zero vector $\psi \in \mathcal{H}$, we define the *ray* containing ψ to be the set:

$$[\psi] = \{\lambda\psi : \lambda \in \mathbb{C}\}.$$

  The set of all rays, called the *projective Hilbert space* $P(\mathcal{H})$, partitions $\mathcal{H}\backslash\{0\}$. The set of physical states $\mathcal{S}$ is in bijection with some subset $\mathcal{P} \subseteq P(\mathcal{H})$, called the set of *physical rays*. We usually set $\mathcal{P} = P(\mathcal{H})$.[1]
- **Postulate III:** The set of observables $\mathcal{O}$ is in bijection with some subalgebra $\mathcal{A} \subseteq H(\mathcal{H})$ of the algebra of Hermitian operators on the Hilbert space.
- **Postulate IV:** (**The Born rule**) If the system $\mathcal{Q}$ is in the physical state corresponding to the ray $[\psi] \in \mathcal{P}$, on a measurement of the observable corresponding to the Hermitian operator $A \in \mathcal{A}$, we see the eigenvalue a^* of A with probability:

$$\mathrm{Prob}(\text{see } a^*) = |([\psi], [\psi_{a^*}])|^2,$$

  where ψ_{a^*} is the projection of ψ onto the eigenspace of A with eigenvalue a^*, and the bracket $(\cdot,\cdot) : P(\mathcal{H}) \times P(\mathcal{H}) \to \mathbb{C}$ is called the *ray product*, defined by:

$$([\psi], [\phi]) = \frac{(\psi, \phi)}{\sqrt{(\psi,\psi)(\phi,\phi)}}.$$

  After the measurement, the system is in the physical state corresponding to the ray $[\psi_{a^*}]$.
- **Postulate V:** Suppose we have two quantum systems, described respectively by the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Then the joint system corresponds to the tensor product Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$.
- **Postulate VI:** Let t be a time coordinate for the quantum system in some frame of reference, and let $|\psi(t)\rangle$ be the state of the system at that time. There exists a distinguished observable H, called the Hamiltonian, which describes the evolution of the state via the *Schrödinger equation*:

$$i\frac{d\,|\psi(t)\rangle}{dt} = H\,|\psi(t)\rangle\,.$$

We should now decide how symmetries should be implemented in the quantum theory. Physically, a natural way to think about a 'symmetry' is: *a reversible transformation of the physical state of a system that leaves all experimental observations the same*. In quantum mechanics, this statement has the following translation into mathematics:

- A transformation of the physical state of a quantum system $\mathcal{Q}$ with corresponding Hilbert space $\mathcal{H}$ is some map

$$S : \mathcal{P} \to \mathcal{P},$$

  where $\mathcal{P}$ is the subset of physical rays in $P(\mathcal{H})$. The requirement that a symmetry is a *reversible* map implies that S should be invertible, and hence a bijection.

- If we scan through the axioms of quantum mechanics, summarised on the previous page, we see that the only actual prediction of the quantum theory is the *Born rule*, which tells us the probability distribution for the outcome of any measurement. Therefore, to ensure that experimental observations remain the same under a symmetry transformation, all we have to do is ensure that this probability distribution remains unaffected - we see that this is indeed fulfilled if we ask a symmetry S to obey:

$$(S[\psi], S[\phi]) = ([\psi], [\phi]),$$

  i.e. we ask any symmetry to *preserve the ray product*.

Hence, a reasonable definition of a symmetry in quantum mechanics might be the following (this definition was introduced by *Eugene Wigner* in 1931):

Definition: Let $\mathcal{Q}$ be a quantum system with associated Hilbert space $\mathcal{H}$. Let $\mathcal{P}$ be the set of physical rays in the projective Hilbert space $P(\mathcal{H})$. A *symmetry* (in the sense of Wigner) is a bijective map of the physical rays $S : \mathcal{P} \to \mathcal{P}$ which preserves the ray product, $(S[\psi], S[\phi]) = ([\psi], [\phi])$.

This is all well and good, but projective Hilbert space $P(\mathcal{H})$ is a pretty nasty place to have to do calculations - it's not even a vector space! We would much rather exploit the vector space structure of $\mathcal{H}$ to do calculations. Therefore, we would like a way of 'projecting down symmetries' from being transformations of projective Hilbert space to being operators on Hilbert space.

A natural way to do this is the following. At the level of projective Hilbert space, a symmetry maps a ray $[\psi] \in \mathcal{P}$ to a ray $S[\psi] \in \mathcal{P}$. If we were working in Hilbert space $\mathcal{H}$ however, we'd have to first pick a *representative* of this ray before we applied the transformation. Suppose we pick the representative $\phi \in [\psi]$. After the transformation, this representative must be sent to some representative of the transformed ray, $S[\psi]$. Hence our transformed representative ϕ' should obey $\phi' \in S[\psi]$.

Hence our desired Hilbert space operator $U : \mathcal{H} \to \mathcal{H}$ should be such that for all $\phi \in [\psi]$, we have $U\phi \in S[\psi]$. This is precisely the notion we want of 'projecting down' the symmetry transformation to an operator on Hilbert space:

Definition: Let $S : \mathcal{P} \to \mathcal{P}$ be a symmetry transformation on the physical rays of a projective Hilbert space $P(\mathcal{H})$. We say that an operator $U : \mathcal{H} \to \mathcal{H}$ on Hilbert space is *compatible* with S if for all rays $[\psi] \in \mathcal{P}$, we have that

$$\phi \in [\psi] \qquad \text{implies} \qquad U\phi \in S[\psi].$$

[1] The case $\mathcal{P} \subsetneq P(\mathcal{H})$ implies having some rays in projective Hilbert space which do not correspond to physical states. This can occur if certain symmetries are present; for example, in a Hilbert space involving both bosonic and fermionic states, superpositions of bosons and fermions are forbidden physically but still exist in the projective Hilbert space. We say that a *superselection rule* is in force, and that the bosonic and fermionic states live in different *superselection sectors* of the Hilbert space. We ignore such subtleties in this course.

Fortunately, there are always operators compatible with symmetry transformations, and they take a very specific form:

Wigner's Theorem: Suppose that the set of physical rays of a quantum system with Hilbert space $\mathcal{H}$ is given by the whole projective Hilbert space $\mathcal{P} = P(\mathcal{H})$. Then for every symmetry $S : \mathcal{P} \to \mathcal{P}$, there exists a compatible operator $U : \mathcal{H} \to \mathcal{H}$ such that U is either:

(i) *linear* and *unitary*. Recall that U is *linear* if $U(\lambda\psi + \eta\phi) = \lambda U\psi + \eta U\phi$ for all $\lambda, \eta \in \mathbb{C}$ and $\psi, \phi \in \mathcal{H}$, and U is *unitary* if $(U\psi, U\phi) = (\psi, \phi)$ for all $\psi, \phi \in \mathcal{H}$.

(ii) *anti-linear* and *anti-unitary*. Recall that U is *anti-linear* if $U(\lambda\psi + \eta\phi) = \lambda^* U\psi + \eta^* U\phi$ for all $\lambda, \eta \in \mathbb{C}$ and $\psi, \phi \in \mathcal{H}$, and U is *anti-unitary* if $(U\psi, U\phi) = (\psi, \phi)^*$ for all $\psi, \phi \in \mathcal{H}$.

If the Hilbert space $\mathcal{H}$ is one-dimensional, then there exists a compatible linear, unitary U *and* a compatible anti-linear, anti-unitary U'.

Proof: Non-examinable. ☐

2: Projective representations and Bargmann's theorem

We are now in a good position to construct a quantum theory which has symmetry group G. By Wigner's theorem, we now know that to implement the group G in the quantum theory every symmetry $g \in G$ should be represented by either a linear, unitary or anti-linear, anti-unitary operator $U(g) : \mathcal{H} \to \mathcal{H}$. If we write $\mathcal{U}(\mathcal{H})$ for the set of operators on the Hilbert space $\mathcal{H}$ that are either linear, unitary or anti-linear, anti-unitary, we see that we must choose an appropriate map:

$$U : G \to \mathcal{U}(\mathcal{H}).$$

However, our choice of U shouldn't be arbitrary. Instead, we should choose U such that we can perform composition of symmetries in the group G or in the space $\mathcal{U}(\mathcal{H})$. This implies that for all states $\psi \in \mathcal{H}$, we must have:

$$U(S_1)U(S_2)\psi = c(S_1, S_2, \psi)U(S_1S_2)\psi,$$

for some complex number $c(S_1, S_2, \psi) \neq 0$, which appears since physical states are rays in Hilbert space. We first notice that this constant must be a phase, which we can prove as follows:

Theorem: The above constant $c(S_1, S_2, \psi)$ must be a phase.

Proof: Note that for both unitary and anti-unitary U we have $(U\psi, U\psi) = (\psi, \psi)$, since in the anti-unitary case we have $(U\psi, U\psi) = (\psi, \psi)^* = (\psi, \psi)$ by conjugate symmetry of the inner product. Hence we have for all $\psi \in \mathcal{H}$:

$$\begin{aligned}
(\psi, \psi) &= (U(S_1)U(S_2)\psi, U(S_1)U(S_2)\psi) && (U(S_1), U(S_2) \text{ are (anti-)unitary})\\
&= (c(S_1, S_2, \psi)U(S_1S_2)\psi, c(S_1, S_2, \psi)U(S_1S_2)\psi) && (\text{definition of } c(S_1, S_2, \psi))\\
&= |c(S_1, S_2, \psi)|^2 (U(S_1S_2)\psi, U(S_1S_2)\psi) && (\text{bilinearity})\\
&= |c(S_1, S_2, \psi)|^2 (\psi, \psi) && (U(S_1S_2) \text{ is (anti-)unitary}).
\end{aligned}$$

It follows that $|c(S_1, S_2, \psi)|^2 = 1$ as required. ☐

Thus, let us write $c(S_1, S_2, \psi) = e^{i\gamma(S_1, S_2, \psi)}$.

The phase $\gamma(S_1, S_2, \psi)$ could in principle depend on the symmetries we are using and the vector ψ we are acting on - however, we can simplify things in the case where the set of physical states is the whole projective Hilbert space:[2]

Theorem: If $\mathcal{P} = P(\mathcal{H})$, the phase $\gamma(S_1, S_2, \psi)$ is independent of the state ψ.

Proof: By unitarity of antiunitarity, for any pair of vectors $\phi, \psi \in \mathcal{H}$ we have:

$$\begin{aligned}\chi((\phi,\psi)) &= (U(S_1)U(S_2)\phi, U(S_1)U(S_2)\psi) = (e^{i\gamma(S_1,S_2,\phi)}U(S_1S_2)\phi, e^{i\gamma(S_1,S_2,\psi)}U(S_1S_2)\psi) \\ &= e^{i(\gamma(S_1,S_2,\psi)-\gamma(S_1,S_2,\phi))}\chi((\phi,\psi)).\end{aligned}$$

where $\chi(z) = z$ or $\chi(z) = z^*$ depending on the unitary or antiunitary nature of the operators (note that if $U(S_1S_2)$ is unitary or antiunitary, then $U(S_1)U(S_2)$ must be unitary or anitunitary respectively).

It follows that whenever $(\phi, \psi) \neq 0$, we must have $e^{i(\gamma(S_1,S_2,\psi)-\gamma(S_1,S_2,\phi))} = 1$. Hence $\gamma(S_1, S_2, \psi) \equiv \gamma(S_1, S_2, \phi)$ (mod 2π) whenever ψ, ϕ are non-orthogonal, and hence γ can be chosen independent of the state.

In the case that ϕ, ψ *are* orthogonal, consider $\psi + \phi \in \mathcal{H}$ (if $\mathcal{P} \neq P(\mathcal{H})$, this vector may not correspond to a physical state, hence the assumption in the theorem). Then:

$$\begin{aligned}e^{i\gamma(S_1,S_2,\psi+\phi)}U(S_1S_2)\psi + e^{i\gamma(S_1,S_2,\psi+\phi)}U(S_1S_2)\phi &= e^{i\gamma(S_1,S_2,\psi+\phi)}U(S_1S_2)(\psi+\phi) \\ &= U(S_1)U(S_2)(\psi+\phi) \\ &= U(S_1)U(S_2)\psi + U(S_1)U(S_2)\phi \\ &= e^{i\gamma(S_1,S_2,\psi)}U(S_1S_2)\psi + e^{i\gamma(S_1,S_2,\phi)}U(S_1S_2)\phi,\end{aligned}$$

by linearity or antilinearity of $U(S_1), U(S_2), U(S_1S_2)$. Now, unitary operators and antiunitary operators both preserve inner products (up to complex conjugation in the latter case), so in particular preserve norms and preserve orthogonality. This if ϕ, ψ are orthogonal, we must have $U(S_1S_2)\phi, U(S_1S_2)\psi$ orthogonal too. Comparing coefficients in the above then, we have:

$$e^{i\gamma(S_1,S_2,\psi+\phi)} = e^{i\gamma(S_1,S_2,\psi)} = e^{i\gamma(S_1,S_2,\phi)},$$

so that $\gamma(S_1, S_2, \psi + \phi) = \gamma(S_1, S_2, \psi) = \gamma(S_1, S_2, \phi)$, modulo 2π. It follows that we can choose $\gamma(S_1, S_2, \psi)$ independent of ψ in all cases. □

It follows that our 'symmetry implementation map' $U : G \to \mathcal{U}(\mathcal{H})$ must obey the general property:

$$U(S_1)U(S_2) = e^{i\gamma(S_1,S_2)}U(S_1S_2).$$

This property is a slight generalisation of what it means to be a representation; we call U a *projective representation*:

Definition: A map $U : G \to GL(\mathcal{V})$, where $\mathcal{V}$ is a complex vector space, is called a *projective representation of the group* G if it obeys the condition:

$$U(g)U(h) = e^{i\gamma(g,h)}U(gh)$$

for all $g, h \in G$. The phase $\gamma(g, h)$ is called a *cocyle*.

[2] In the case where $\mathcal{P} \subsetneq P(\mathcal{H})$, then it is possible to show that the Hilbert space breaks up into a direct sum of spaces, called *superselection sectors*; we can then show that $\gamma(S_1, S_2, \psi)$ depends only on S_1, S_2 and the superselection sector to which ψ belongs.

The above discussion has led us to conclude that classifying the ways that a group G can be implemented in a quantum theory is the same task as classifying (unitary and anti-unitary) projective representations of the group G.

This doesn't seem very useful, because throughout the course we only discuss *representations* rather than *projective representations*. However, there is an important theorem that tells us how we can 'deprojectivise' (i.e. pass from a projective representation to an ordinary representation) projective representations of Lie groups; this theorem is called *Bargmann's theorem*. First, we must define some new mathematical terms:

Definition: Let M and C be topological spaces. We say that C is a *covering space* for M if there exists a continuous, surjective map $p : C \to M$, such that for all open sets $U \subseteq M$, the preimage $p^{-1}(U)$ is the disjoint union of open sets in C, each of which is homeomorphic to U via p. We call p the *covering map* for the covering space.

A *universal covering space* for M is a covering space C which is simply-connected. It can be shown that universal covering spaces are unique up to homeomorphism.

The *universal covering group* of a Lie group G is the universal covering space $\tilde{G}$ of G viewed as a topological space. It can be shown that $\tilde{G}$ is also a Lie group.

With these terms, we can make precise the sense in which we wish to 'deprojectivise' the projective representation:

Definition: Let $\hat{U} : G \to GL(\mathcal{V})$ be a projective representation of the group G on the complex vector space $\mathcal{V}$, and let $\tilde{G}$ be the universal covering group of G, with covering map $p : \tilde{G} \to G$.

Suppose that $U : \tilde{G} \to GL(\mathcal{V})$ is an ordinary representation of the universal covering group $\tilde{G}$ on the complex vector space $\mathcal{V}$. We say that $\hat{U}$ is a *lift* of U to the universal covering group if we have:

$$U = \hat{U} \circ p.$$

This final condition ensures that we can 'project back' to the projective representation using the covering map.

We are now ready to state the theorem:

Bargmann's Theorem: Let $\hat{U}$ be a projective, unitary representation of a Lie group G. Then $\hat{U}$ can be lifted to an ordinary, unitary representation U of the universal covering group $\tilde{G}$.

Proof: Beyond the scope of the course. □

Note that the theorem is about unitary representations, but usually we can just add in the anti-unitary elements of the representation as an afterthought (e.g. in representations of the Poincaré group, the representations are generated completely by unitary elements and a single anti-unitary element related to reversing the flow of time).

This theorem has an exceptionally important application in quantum mechanics: if we wish to implement the symmetry group G in quantum mechanics, it is sufficient to classify the *ordinary*, *non-projective* unitary (or anti-unitary) representations of its universal covering group $\tilde{G}$. This is essentially what we shall do in the rest of the course, when we consider implementing rotations $SO(3)$ in quantum mechanics (with universal cover $SU(2)$) and the Lorentz group $SO^+(1,3)$ in quantum mechanics (with universal cover $SL(2,\mathbb{C})$).[3]

[3] More generally, the universal covering groups of $SO(p), SO(p,q), SO^+(p,q)$ are called the *spin groups* $\mathrm{Spin}(p), \mathrm{Spin}(p,q), \mathrm{Spin}^+(p,q)$ (so that $\mathrm{Spin}(3) \cong SU(2)$ and $\mathrm{Spin}^+(1,3) \cong SL(2,\mathbb{C})$) - the reason for the name will become apparent later in the course, when we see that transitioning to the universal cover forces us to introduce *spin* in quantum mechanics.

3: Applications of symmetry in QFT

There are numerous applications of symmetry in QFT, but here are a few of the most important ones:

- CLASSIFYING FREE PARTICLES. The story we have told throughout this handout has a very important consequence when the group G we wish to implement in the quantum theory is the group of *spacetime symmetries*. Let's suppose that

$$U : \tilde{G} \to \mathcal{U}(\mathcal{H})$$

is an ordinary representation of the universal covering group $\tilde{G}$ of the spacetime symmetry group G, consisting of unitary and anti-unitary operators on some Hilbert space $\mathcal{H}$. An interesting question we might ask is: what is the interpretation of an *irreducible* representation U?

Recall that spacetime symmetries correspond to changing observers. In particular, this means that $U(g)\psi$ should be interpreted as a 'rotated version of ψ' or a 'translated version of ψ', etc. In particular, any representation $\mathcal{H}$ (since it has an invariant inner product, as U is unitary) can be decomposed into irreducible representations:

$$\mathcal{H} = \bigoplus_{\alpha} n_\alpha \mathcal{H}_\alpha$$

where the irreducible representations $\mathcal{H}_\alpha$ correspond to 'states related by changing observers'. On the contrary, we cannot relate $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$ for $\alpha \neq \beta$ by some change of observers.

Wigner realised that this meant we could consider the irreducible representations $\mathcal{H}_\alpha$ to be the 'Hilbert spaces of free particles'. This is because each of the $\mathcal{H}_\alpha$ has different transformation properties under the symmetry group's universal cover $\tilde{G}$ (otherwise they would be the same as irreducible representations), and hence can be distinguished from one another physically, but states within $\mathcal{H}_\alpha$ should be considered as 'viewing a free particle state from different reference frames'. The reason we consider these particles 'free' is because the symmetry group G of spacetime usually include time translations, so we have to assume that the particle is doing nothing - it looks the same at any time.

In the case of the Poincaré group, which is the symmetry group of Minkowski spacetime, we find that the irreducible unitary representations are labelled by two numbers: an arbitrary real number $m \in \mathbb{R}$, which we interpret as the mass of the particle, and a non-negative integer s, which we interpret as the spin (or *helicity* in some cases - see later on) of the particle. Hence the fundamental properties of mass and spin pop out of the theory magically! We will see this later in the course when we study irreducible representations of the Poincaré group (this classification is called the *Wigner classification*).

The key point to take away from this intuition is:

Definition: An irreducible representation of the universal covering group $\tilde{G}$ of the spacetime symmetry group G is called a *free particle* (in the sense of Wigner).

We can use this definition to construct fully-fledged interacting theories:

- CONSTRUCTION OF QUANTUM FIELDS. States with more than one free particle are to be found in the *tensor product spaces* $\mathcal{H}_{\alpha_1} \otimes \ldots \otimes \mathcal{H}_{\alpha_n}$. From these spaces, we can form a new Hilbert space called *Fock space*, given by:

$$F = \mathbb{C} \oplus \left(\bigoplus_{\alpha} \mathcal{H}_\alpha \right) \oplus \ldots \oplus \left(\bigoplus_{\alpha_1, \ldots, \alpha_n} \mathcal{H}_{\alpha_1} \otimes \ldots \otimes \mathcal{H}_{\alpha_n} \right) \oplus \ldots$$

A Fock space is supposed to capture the notion that we could have no particles in our system, or we could have one particle in our system, or we could have n particles in our system, etc.

We can move between the n particle space and the $n+1$ particle space using *creation* and *annihilation* operators. Let's focus on the case that there is a single irreducible representation of our spacetime symmetry group, call it $\mathcal{H}$, and suppose that this space is spanned by the basis states $\mathcal{H} = \{|p\rangle\}$ where p is some label. Then Fock space takes the form:

$$F = \mathbb{C} \oplus \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H}) \oplus \cdots .$$

Writing the n-fold tensor product of $\mathcal{H}$ as $\mathcal{H}^n$, we define the creation and annihilation operators via:[4]

$$a^\dagger_{\tilde{p}} : \mathcal{H}^n \to \mathcal{H}^{n+1}, \qquad a^\dagger_{\tilde{p}} |p_1, ..., p_n\rangle = |p_1, ..., p_n, \tilde{p}\rangle ,$$

with the appropriate extension by linearity, and

$$a_{\tilde{p}} : \mathcal{H}^{n+1} \to \mathcal{H}^n, \qquad a_{\tilde{p}} |p_1, ..., p_n\rangle = \begin{cases} 0 & \text{if } p_i \neq \tilde{p} \text{ for any } i; \\ |p_1, ..., p_{i-1}, p_{i+1} ... p_n\rangle & \text{if } p_i = \tilde{p}, \end{cases}$$

again with the appropriate extension by linearity.

It is possible to show that any operator on Fock space can be written in terms of creation and annihilation operators. In particular, some Hamiltonian for the system H (which could include interactions where particles are created or destroyed) must be expressible in terms of creation and annihilation operators.

It turns out that under the further assumption that such a H is a *local* function (i.e. can be written as an integral of some operator, representing the Hamiltonian density, over spacetime) that H must be expressed only in terms of specific linear combinations of creation and annihilation operators, namely *quantum fields*. However, this becomes quite technical, and we will not discuss the details in this course - see Weinberg's *Quantum Theory of Fields Volume I* for further information.

- GAUGE SYMMETRY. As a last application we consider *gauge symmetry*, but in even vaguer terms than the previous point (a proper discussion takes us too far off track from this course). If we take the spacetime symmetry group G to be the Poincaré group in the above, then one of the possible irreducible representations we find is a *spin-1 massless particle*. However, there is a complication in this case when we want to write down an associated quantum field - it turns out that any quantum field we wish to write down in this case is actually *undetermined*. The freedom in the description is called the *gauge redundancy* of the field.

  If we choose to have a spin-1 massless particle in our theory then (which we must, because they exist in Nature!), we will be working with a *gauge theory*. Choosing the way that the spin-1 massless particle self-interacts determines the *gauge group* of the theory, which should be selected to correspond to Nature.

So symmetry is very important - it determines a huge amount about the structure of a quantum theory! These points should motivate the more detailed exposition we will cover later in the course.

[4] Actually, slightly more care is necessary here, because we wish to identify the states $|p_1, p_2, ..., p_n\rangle$ and $|p_2, p_1, ..., p_n\rangle$ for example. This requires a discussion of *particle statistics*, which can lead to phases being necessary in the definitions of the creation and annihilation operators. We know we've chosen the right or the wrong phase when we try to write down associated quantum fields and things go wrong - this is the famous *spin statistics theorem*, but it is not part of the main narrative of this course. The definitions we have written down for the creation and annihilation operators only work in the bosonic case; a more detailed treatment is given in Weinberg's *Quantum Theory of Fields Volume I*.