Part IB: Electromagnetism
Solving Electrostatics vs Magnetostatics

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Electrostatics

The electric field $\mathbf{E} : \mathbb{R}^3 \to \mathbb{R}^3$ produced by a static charge distribution $\rho : \mathbb{R}^3 \to \mathbb{R}$ is given by the equations of *electrostatics*:

Equations of electrostatics:

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0 \qquad \text{(Gauss' law)}$$

$$\nabla \times \mathbf{E} = \mathbf{0}$$

We can recast the first equation in integral form by integrating over an arbitrary volume V and applying the divergence theorem:

$$\int_{\partial V} \mathbf{E} \cdot \mathbf{dS} = Q/\epsilon_0,$$

where ∂V is the boundary of V and Q is the total charge contained in V.

Highly symmetric solutions. When the charge distribution ρ possesses a lot of symmetry, one can usually argue that the functional form of the electric field must reduce to a simple form (using the fact that the electric field transforms as a *vector* under translations, rotations and reflections). In this case, we can use the differential or integral forms of Gauss' law from above to obtain the electric field easily.

General solution. If the charge distribution ρ is not very symmetric, we can still solve the equations. To obtain a general solution:

(1) Note $\nabla \times \mathbf{E} = \mathbf{0}$ implies that $\mathbf{E} = -\nabla\Phi$ for some Φ.

(2) Substituting $\mathbf{E} = -\nabla\Phi$ into Gauss' law, we obtain $\nabla^2\Phi = -\rho/\epsilon_0$. This is *Poisson's equation*.

(3) Using the free-space Green's function solution to Poisson's equation from Part IB Methods, we see that:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\, d^3\mathbf{x}'.$$

Taking the gradient of this formula, we obtain the electric field.

Magnetostatics

The magnetic field $\mathbf{B} : \mathbb{R}^3 \to \mathbb{R}^3$ produced by a static current distribution $\mathbf{J} : \mathbb{R}^3 \to \mathbb{R}^3$ is given by the equations of *magnetostatics*:

Equations of magnetostatics:

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \qquad \text{(Ampère's law)}$$

We can recast the second equation in integral form by integrating over an arbitrary surface S and applying Stoke's theorem:

$$\oint_{\partial S} \mathbf{B} \cdot \mathbf{dx} = \mu_0 I$$

where ∂S is the boundary of the surface S, and I is the current passing through the surface S.

Highly symmetric solutions. When the current distribution $\mathbf{J}$ possesses a lot of symmetry, one can usually argue that the functional form of the magnetic field must reduce to a simple form (using the fact that the magnetic field transforms as a *pseudovector* under translations, rotations and reflections). In this case, we can use the differential or integral forms of Ampère's law from above to obtain the magnetic field easily.

General solution. If the current distribution $\mathbf{J}$ is not very symmetric, we can still solve the equations. To obtain a general solution:

(1) Note $\nabla \cdot \mathbf{B} = 0$ implies that $\mathbf{B} = \nabla \times \mathbf{A}$ for some $\mathbf{A}$.

(2) Substituting $\mathbf{B} = \nabla \times \mathbf{A}$ into Ampère's law, and expanding using a standard identity from vector calculus, we obtain:

$$\nabla^2\mathbf{A} - \nabla(\nabla \cdot \mathbf{A}) = -\mu_0 \mathbf{J}. \qquad (*)$$

(3) Using the *gauge freedom* $\mathbf{A} \mapsto \mathbf{A}' = \mathbf{A} + \nabla\chi$, we can WLOG choose $\mathbf{A}$ such that $\nabla \cdot \mathbf{A} = 0$. Using the Green's function, we have:

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{x})}{|\mathbf{x} - \mathbf{x}'|}\, d^3\mathbf{x}'.$$

Taking the curl of this formula, we obtain the magnetic field.